https://www.linkedin.com/feed/update/urn:li:activity:6635914108917043200

Find all real solutions of the following system of equations $x_1x_2+1=4x_2$, $x_2x_3+1=x_3, x_3x_4+1=4x_4, \dots, x_{2019}x_{2020}+1=4x_{2020}$, $x_{2020}x_1+1=x_1$.

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We will solve the original problem in following more general version:

For any natural $m \ge 2$ find all real solutions of the following system of equations

$$(\mathbf{1}) \quad \begin{cases} x_{2k-1}x_{2k}+1=4x_{2k}, k=1,2,\dots,m \\ x_{2k}x_{2k+1}+1=x_{2k+1}, k=1,2,\dots,m-1 \\ x_{2m}x_1+1=x_1 \end{cases}.$$

We have (**1**)$\Leftrightarrow \begin{cases} x_{2k}=h(x_{2k-1}), k=1,2,\dots,m \\ x_{2k+1}=f(x_{2k}), k=1,2,\dots,m-1 \\ x_1=h(x_{2m}) \end{cases}$,where $h(x):=\dfrac{1}{4-x}$,

$f(x):=\dfrac{1}{1-x}$ and since $f(h(x))=\dfrac{1}{1-\dfrac{1}{4-x}}=\dfrac{x-4}{x-3}=1-\dfrac{1}{x-3}$ then (**1**)$\Leftrightarrow$

$$(\mathbf{2}) \quad \begin{cases} x_{2k+1}=1-\dfrac{1}{x_{2k-1}-3}, k=1,2,\dots,m-1,\ x_1=1-\dfrac{1}{x_{2m-1}-3} \\ x_{2k}=\dfrac{1}{4-x_{2k-1}}, k=1,2,\dots,m \end{cases}.$$

Denoting $t_k:=x_{2k-1}-3, k=1,2,\dots,m$ we can equivalently rewrite the system (2) as

$$(\mathbf{3}) \quad \begin{cases} t_{k+1}=-2-\dfrac{1}{t_k},\ k=1,2,\dots,m-1, t_1=-2-\dfrac{1}{t_m} \\ x_{2k-1}=t_k+3,\ x_{2k}=\dfrac{1}{4-x_{2k-1}}, k=1,2,\dots,m \end{cases}.$$

Consider a sequence $(t_n)_{n\in\mathbb{N}}$ defined by $t_1=t\neq 0, t_{n+1}=-2-\dfrac{1}{t_n}, n\in\mathbb{N}$.

Let $p_n:=t_1t_2\dots t_n, n\in\mathbb{N}$ and $p_0=1$. Then $p_1=t,\ t_n:=\dfrac{p_n}{p_{n-1}}, n=1,2,\dots,$ and

$t_{n+1}=-2-\dfrac{1}{t_n}$ becomes $\dfrac{p_{n+1}}{p_n}=-2-\dfrac{p_{n-1}}{p_n} \Leftrightarrow p_{n+1}+2p_n+p_{n-1}=0 \Leftrightarrow$

$p_{n+1}+p_n=-(p_n+p_{n-1}), n\in\mathbb{N} \Leftrightarrow p_{n+1}+p_n=(-1)^n(p_1+p_0)=(-1)^n(t+1), n\in\mathbb{N} \Leftrightarrow$

$\dfrac{p_{n+1}}{(-1)^n}=\dfrac{p_n}{(-1)^{n-1}}+t+1 \Leftrightarrow \dfrac{p_n}{(-1)^{n-1}}=\dfrac{p_0}{(-1)^{0-1}}+n(t+1)=-1+n(t+1)=$

$nt+n-1, n\in\mathbb{N}$. Hence, $p_n=(-1)^{n-1}(nt+n-1), n\in\mathbb{N}$ and, therefore,

$t_n=\dfrac{p_n}{p_{n-1}}=-\dfrac{nt+n-1}{(n-1)t+n-2}, n\in\mathbb{N}$.

Coming back to the system (**3**) (since $t_1=-2-\dfrac{1}{t_m}=t_{m+1}$) we obtain

$$t=-\frac{(m+1)t+m}{mt+m-1} \Leftrightarrow (mt+m-1)t+((m+1)t+m)=0 \Leftrightarrow m(t+1)^2=0 \Leftrightarrow t=-1.$$

Since $t=-1$ is fixed point of $-2-\dfrac{1}{t}$ then $\begin{cases} t_{k+1}=-2-\dfrac{1}{t_k}, k=1,2,\dots,m-1 \\ t_1=-2-\dfrac{1}{t_m} \end{cases} \Leftrightarrow$

$t_1=t_2=\dots=t_m=-1$ and, therefore, $x_1=x_3=\dots=x_{2m-1}=2, x_2=x_4=\dots=x_{2m}=\dfrac{1}{2}$

is only solution of the system (**1**) .